Economic development and sustainability in a two-sector model with variable population growth rate

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Outline

- Long-run growth and recent development
- The model
- Long-run sustainability conditions
- Tax rate and sustainability
- Sustainable Steady State
- Conclusions
Long-run growth: a brief excursus

- Solow-Swan model (1956)
- Generalized models: Hall and Taylor, Mankiw, Romer, etc.
- Dasgupta (1996) introduces environmental resources
- Tran-Nam (2001) inspired the present work
Beyond Tran-Nam’s model

- It was shown if human activities produce a net beneficial effect on the environment then the economy will converge to a unique and stable steady state.
- A natural question: what the impact of changes in the population growth rate would be?
- We are going to analyse the consequences of relaxing the assumption of constant population growth rate.
Fundamental tools and remarks

- Following Guerrini (2006), Ferrara and Guerrini (2008), we consider the labor growth rate to be variable over time and controllable (upper and lower limits).
- Natural capital stock is modeled as a renewable resource.
- Environmental capital as a stock of measurable in some constant quality units.
- The change in the stock capital depends on its evolution, production and consumption externalities and environment programs.
Research focus

- We find out the long-run sustainability depends crucially on human activities and on the stock of the environment.
- We discover the economy is sustainable if the human activities have a net zero effect on the environment and the stock grows or remains unchanged over time.
- By modelling the natural capital stock as a renewable resource, we have that damages done to the environment production and consumption externalities are reversible and can be corrected by environment programs.
The model

- We consider a closed economy in continuous time where a homogenous good is produced according to a technology (three inputs: physical capital, natural capital and labor), i.e.:

\[ Y(t) = K(t)^\eta E(t)^\theta L(t)^{1-\eta-\theta}, \]

\[ \eta, \theta, \eta + \theta \in (0,1) \]
The model

Based on feature of constant returns of scale, $Y$ may be written in terms of capital per workers as

$$y = k^\eta e^\theta$$

where $y=Y/L$ is the per capita capital output (the same for $k$ and $e$)
The model

Output is assumed to be used for consumption $C$, for savings $S$, or spent to maintain or improve the environment. The national accounting is given by:

$$Y = C + S + T$$

where $S$ and $T$ stand for saving and tax, respectively.
Basic tools

- Tax revenue: \( T=\tau Y \), with \( \tau \in (0,1) \)
- Consumption: \( C=a(Y-T) \), with \( a \in (0,1) \)
- \( \tau \) = tax rate, \( a \) = marginal propensity to consume
- Depreciation rate: \( \delta K \), where \( K \) is the capital stock
Accumulation process

- Net increase in capital stock:
  \[ \dot{K} = I - \delta K \]

- Capital accumulation process (equ. 1):
  \[ \dot{K} = (1 - a)(1 - \tau)Y - \delta K \]
Dynamics of the physical capital stock

By differentiating $K/L$ with respect to time and substituting equation (1), we obtain:

$$
\dot{k} = (1 - a)(1 - \tau)k^\eta e^\theta - \left( \delta + \frac{\dot{L}}{L} \right)k
$$
Population growth rate

- Classical assumption: population=labor force
- Growth:
  \[ \dot{L} = nL \Rightarrow L(t) = L(0)e^{nt} \]
- Contrary to the standard literature we assume that \( \frac{dL}{dt} \) at any moment is a function of the population size
Variable population

- We may write:

\[ \dot{L} = L \ln(L) \]

where \( n(L) \) is a function of \( L \).

- \( n(L) \) is controllable subject to be between upper and lower limits:

\[ 0 \leq n(L) \leq M, \quad \lim_{t \to \infty} n(L) = 0 \quad n_L(L) < 0 \]
Variable population

- In particular, we have:

\[ L(0) \leq L(t) \leq L(0)e^{Mt}, \forall t \]

- Furthermore, let us assume \( L(0)=1, \lim_{t \to \infty} L=L_\infty < \infty \) and there exists a unique value \( L_* \neq 0 \) s.t. \( n(L)=0 \)

- An example of population growth rate can be a logistic map (see Verhulst)
Remark

- $L_\infty < \infty$ yields $n(L_\infty) = 0$.
- In fact, let $\varphi : [x_0, +\infty) \to \mathbb{R}$ be a differentiable function s.t. there exist (finite or infinite) the limits $\lim_{x \to +\infty} \varphi(x) = l$, $\lim_{x \to +\infty} \varphi'(x) = n$. If $l$ is finite, then $n = 0$.
- Note that $n(L_\infty) = 0$, and so $L_\infty = L_*$.
Regarding the environmental stock $E$ its evolution can be described by:

$$\dot{E} = \alpha E + \phi T - \beta Y - \gamma C,$$

where $\alpha, \phi, \beta$ and $\gamma$ are some constants.
More on

- **No human activity**: $E$ changes over time at exponential rate $\alpha$ ($>0 \Rightarrow E$ grows, $=0 \Rightarrow E$, $<0 \Rightarrow E$ decays autonomously)

- **human activity**: depletion of $\beta$ units of $E$ for every unit of good produced, each unit of consumed good depletes $\gamma$ units of $E$

- Environmental programs, funded by $T$, generate $\phi$ units of $E$ per unit of the tax spent.
Environmental stock dynamics

- The general equation can be expressed in per capita terms:

\[
\dot{E} = \alpha E + \left[ (\phi + \gamma_a)\tau - (\beta + \gamma_a) \right] Y
\]

- By differentiating \( E \) per worker with respect to time and by replacing the obtained equations, we get

\[
\dot{e} = \left[ (\phi + \gamma_a)\tau + (\beta + \gamma_a)k^n \theta - [n(L) - \alpha] \right] e
\]
Model’s economy

- Setting \( A=(1-a)(1-\tau)>0, \) \( B=(\phi+\gamma a)\tau-(\beta+\gamma a) \) the model’s economy is described by:

\[
\begin{align*}
\frac{dk}{dt} &= Ak \eta e^\theta - [\delta + n(L)]k \\
\frac{de}{dt} &= Bk \eta e^\theta - [n(L) - \alpha]e \\
\frac{dL}{dt} &= Ln(L)
\end{align*}
\]

- Given \( k_0=k(0)>0, \) \( e_0=e(0) > 0, \) this Cauchy problem has a unique solution \((k(t), e(t), L(t))\), defined on \([0,\infty)\)
Long-run sustainability conditions (L-r.s.c)

- An economy is said to be long run sustainable so long as per capita consumption $c$ equals or exceeds a given subsistence consumption level $k(t)e(t)$ must be positive. If we consider $e(t)$ as a private good, we have that L-r.s.c. require, given a minimum life-sustaining level $\bar{e}, \forall t$

\[ \bar{c} > 0, \text{i.e.} \quad k^n e^\theta \geq \bar{c}/[a(1-\tau)], \forall t \]

- $k(t)$ and $e(t)$ must be positive. If we consider $e(t)$ as a private good, we have that L-r.s.c. require, given a minimum life-sustaining level $\bar{e}, \forall t$

\[ \bar{e} > 0 \quad e(t) \geq \bar{e}, \forall t \]
Results

- In general, there is no guarantee that the stock of physical capital as well as the stock of natural capital will remain positive as time grows indefinitely large.
- In this paper we show this depends crucially on the sign of $B$
- We remember that $B = (\phi + \gamma a) \tau - (\beta + \gamma a)$
The crucial role of $B$

**Proposition 1.**

Let $B=0$.

i) For all $t$, the natural capital is described by

$$e(t) = e_0 \exp(\alpha t)L(t)^{-1},$$

as $t$ grows to infinity, $e(t)$ decreases monotonically to 0 or to $e_0 L_{-1}$ if $\alpha<0$ or $\alpha=0$, respectively, while it diverges to $+\infty$ if $\alpha>0$. 
The crucial role of $B$

ii) For all $t$, the physical capital is described by

$$k(t) = \left[ \exp(\delta t) L(t) \right]^{-1} \left[ k_0^{1-\eta} + Ae^\theta g(t) \right]^{1/(1-\eta)}$$

where

$$g(t) = \int_0^t \exp\{[\alpha \theta + (1-\eta)\delta]t\} L(t)^{1-(\eta+\theta)} dt.$$

In the L-r, $k(t)$ converges to 0 or to $\left[ Ae^\theta / (1-\eta) \delta L_\infty^\theta \right]^{1/(1-\eta)}$ if $\alpha<0$ or $\alpha=0$, respectively, while it diverges to $+\infty$ if $\alpha>0$
Results

Proposition 2.

Let $B < 0$. Then $e(t)$ is monotone decreasing. As $t$ grows to infinity, $e(t)$ converges to $0$ if $\alpha < 0$, it does not diverge to infinity if $\alpha = 0$, its behavior is unknown if $\alpha > 0$; $k(t)$ does not converge to zero no matter who is $\alpha$. Let $B > 0$. Then nothing can be concluded about the long run behavior of the functions $e(t)$ and $k(t)$.
Results

Theorem 1.
If human activity have a net zero or negative effect on the environment in every time period and the stock of the environment decays autonomously over time, i.e. \( B < 0 \) and \( \alpha < 0 \), then the economy is unsustainable in the long run. If human activities have a net zero effect on the environment and the stock of \( E \) grows or remains unchanged autonomously over time, i.e. \( B = 0 \) and \( \alpha \geq 0 \), then the economy is sustainable in the long run.
Remarks

- In case of constant population growth
  \[
  \frac{\dot{L}}{L} = n
  \]
  and the hypothesis \( n > \alpha \), Tran-Nam showed that the economy is always unsustainable in the L-r if \( B \leq 0 \). Moreover a necessary condition to get a sustainability is that \( B > 0 \) i.e. if human activities produce a net beneficial effect on the environment in every period.

- We obtain very different results starting from an other hypothesis (variable population growth rate)
Tax rate and sustainability

A sufficient condition for the economy to be sustainable in the L-r. is that $B=0$ and $\alpha \geq 0$, while a necessary condition is provided by $B<0$ and $\alpha \geq 0$, or $B>0$ and $\alpha$ arbitrary.

Mathematically these conditions can be translated as follows.
Tax rate and sustainability

Lemma 1

i) For any given tax rate $\tau \in (0,1)$, $B(\geq, <) 0$ iff $a (\leq, >)$
   $$(\phi \tau - \beta)/(1-\tau)\gamma;$$

ii) For any given MPC $a \in (0,1)$, $B (\geq, <) 0$ iff $\tau (\geq, <)$
    $$(a\gamma+\beta)/(a\gamma+\phi);$$

iii) $$(\phi \tau - \beta)/((1-\tau)\gamma \leq 0$ iff $\tau \leq \beta/\phi;$$

iv) $$(\phi \tau - \beta)/((1-\tau)\gamma \in (0,1)$$ iff $\beta/\phi < \tau < (\beta+\gamma)/(\phi+\gamma);$  

v) $$(\phi \tau - \beta)/((1-\tau)\gamma \geq 1$$ iff $\tau \geq (\beta+\gamma)/(\phi+\gamma).$$
Tax rate and sustainability

**Lemma 2.**
If $B \geq 0$, then $\phi > \beta$, while if $B < 0$ the relationship between $\phi$ and $\beta$ is undetermined.

**Proposition 3.**
Let $B > 0$ ($\alpha$ arbitrary). For any given tax rate $\tau \in (0,1)$, the set of sustainable MPCs is empty if $\tau \leq \beta/\phi$, it is the interval $(0, (\phi \tau - \beta)/(1 - \tau) \gamma)$ if $\beta/\phi < \tau < (\beta + \gamma)/(\phi + \gamma)$, it is the interval $(0, 1)$ if $\tau \geq (\beta + \gamma)/(\phi + \gamma)$.

.....And so on for $B < 0$ and $B = 0$
Remarks

Some interesting things can be derived from the Proposition 3. For example if \( B > 0 \), we deduce that an increase in the tax rate in the relevant range widens the choice of sustainable MPCs, while a decrease in the tax rate narrows this choice. It is clear that more resources are spent to repair the environment, then, keeping the economy sustainable, a larger fraction of the remaining output is available for consumption and viceversa.
Sustainable Steady State

- A steady state of a sustainable economy is defined as a situation in which the growth rates of the per capita physical capital, the per capita natural capital and the labor growth rate are equal to zero.
- In studying the steady states of our sustainable economy, we will confine our analysis to interior steady states only.
Sustainable Steady State

Proposition 4.

1. No steady states of model’s economy introduced exist if \( B=0 (\alpha>0) \), if \( B<0 (\alpha=0) \), or \( B>0 (\alpha \geq 0) \).
2. There is a unique steady state if \( B<0 (\alpha>0) \), or if \( B>0 (\alpha<0) \):

\[
(k^*, e^*, L^*) = (\omega, -(\delta B / \alpha A) \omega, L^*)
\]

where

\[
\omega = \left[\left(\frac{A}{\delta}\right)^{1-\theta} \left(-\frac{B}{\alpha}\right)^{\theta}\right]^{1/[1-(\eta+\theta)]}
\]
Sustainable Steady State

3. There are infinite steady states if $B=0$ ($\alpha=0$):

$$(k_*, e_*, L_*) = \left( k, \left[ (\delta / A)k^{1-\eta} \right]^{1-\theta}, L_* \right), \forall k > 0$$
Sustainable Steady State

Theorem 2.

If $B=0$ ($\alpha=0$), every steady state equilibrium is unstable.
If $B<0$ ($\alpha>0$), the unique steady state equilibrium is a saddle with a two dimensional stable manifold. If $B>0$ ($\alpha<0$), the unique steady state equilibrium is a stable node.
Remarks

- In case of a constant population growth rate, Tran-Nam showed that, if human activities produce a net beneficial effect on the environment, then the economy will converge to a unique and stable steady state.
Conclusions

- By this work we presented an effort to incorporate natural capital into Solow-Swan Model under the assumption of a variable population growth rate;
- we find out that the economy is sustainable in the l-r. if human activities have a net zero effect on the environment and the stock of E grows or remains unchanged autonomously over time;
- It is unsustainable if human activities have a net zero or negative effect on E and the stock decays over time;
- for any given tax rate (or MPC), we derive the set of sustainable MPCs (or tax rates);
- we examine the non-trivial steady states of a sustainable economy.
Main references


Thanks for your attention